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[This is a translation of an article written by R. Ye. Krichevskiy in Problemy Kibernetiki (Problems of Cybernetics), No 2, Moscow, 1959, pages 123-138.]

In the theory of controlling systems and in mathematical logic we frequently encounter the problem of the simplest expression or realization of the given function by means of constructions of one or another type, for example, by means of iterations (superpositions) of some basic functions or by means of controlling systems. The construction is usually ascribed an index of simplicity which sapidly expresses its complexity, for example, the number contacts in the contact circuit or the number of letters in the formula; the simplest of the constructions which expresses the function is that which has a minimum index of simplicity.

A substantial characteristic of the mass of the functions of D_h is the number $L(D_h)$---the upper edge of the indexes of the simplest constructions which express the functions of D_h; this number was, for the first time, brought up by Riordan and Schannon [1], Schannon [2]. Many works deal with the evaluation of $L(D_h)$ in the concrete selection of the mass D_h, the mass of realizing constructions and the index of simplicity; we wish to mention the papers by Schannon [2], Riordan and Schannon [1], Lupanov [3], Yablonskiy [4], Povarov [5, 6]. In the work [3], the constructions of a very general type are examined as the constructions which express the given function.

We shall examine the "superpositions of elementary objects" as the constructions. The concept of the superposition of elementary objects is introduced as a generalization of the concept of the superposition of basic functions and makes it possible to obtain uniformly results which pertain to the theory of circuits and to the algebra of logic. From the index of simplicity we require the fulfillment of natural limitations (see p. 1. 3 (The first figure indicates the number of the paragraph; the second, the number of the point.). Under these conditions, the lower evaluation for $L(D_h)$ is obtained; we wish to note that the Lupanov method would give here a much coarser evaluation because Lupanov examined constructions of a more general type than in our case.

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In the theory of controlling systems and in mathematical logic we frequently encounter the problem of the simplest expression or realization of the given function by means of constructions of one or another type, for example, by means of iterations (superpositions) of some basic functions or by means of controlling systems. The construction is usually ascribed an index of simplicity which sapidly expresses its complexity, for example, the number contacts in the contact circuit or the number of letters in the formula; the simplest of the constructions which expresses the function is that which has a minimum index of simplicity.

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Determination 1 (inductive). Any symbol from x is called superposition of class 0.

2. Let $L_k(A_1, \ldots, A_s)$ be called the superposition of class n if $s = s_k$ and all A_i — superpositions of a class less than n, moreover, the maximum of the classes of the superpositions A_i, $i = 1, \ldots, s$, is equal to $n-1$.

Superposition the class of which is not less than 1 is called superposition of elementary objects of σ_0. The mass of superpositions of the objects is designated by σ. Examples of the superposition of class 1 are, for the proper slection of the masses $x = \{x_\mu\}$ & $L = \{L_k\}$:

$$L_1(x_1, x_2, x_1, x_3);\ L_2(x_1, x_2, x_3).$$

An example of superposition of class 2 is $L_1(L_1(x_1, x_2, x_1, x_2), x_1, x_2)$.

1.2. Geometric illustration. Henceforth, it will be convenient to utilize the geometric interpretation of superposition. We shall determine inductively the image of superposition, its root, and its end segments. We shall consider an s_k-place star as the image of an s_k-place superposition of class 1 $L_k(x_1, \ldots, x_{s_k})$; i. e., the combination of the s_k+1 segments which emanate from one point, one of which, called the root, is separated, while the remaining, called ends, are numbered in a definite order. We shall compare the root with the symbol L_k and the i-th segment—with the symbol x_i (Fig 1).

Let there now be the superposition of elementary objects of n-th class $L_k(A_1, \ldots, A_{s_k})$, where A_i are the superpositions of the classes not higher than $n-1$, $i = 1, \ldots, s_k$. Let us assume that for the superposition A_i,

$$A_i = L_{k_i}(A_{i(1, i)}, \ldots, A_{j(i, k_i)})$$

a geometric image has already been constructed and that the symbol L_{k_i} corresponds to the root of it and the symbols x_μ encountered in the superposition of A_i correspond to the end segments.

In order to construct the giometric image $L_k(A_1, \ldots, A_{sk})$ we shall identify the i-th end segment of the star, corresponding to $L_k(x_1, \ldots, x_{s_k})$ with the root of the image of A_i, and for the resulting segment we shall accordingly place the symbol L_{k_i}, while the remaining symbols correspond to the previous segments. The root of the image $L_k(A_1, \ldots, A_{s_k})$ will be called the root of the star $L_k(x_1, \ldots, x_{s_k})$, while all the remaining end segments of the images $A_1, \ldots, A_{s_k}$ will be called end segments.

Fig 1

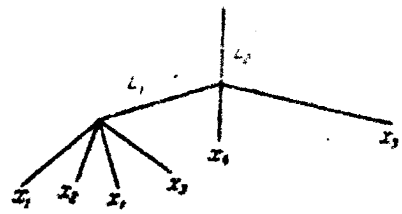

Fig 2

Thus, in the geometric image of the object $L_k(A_1, \dots, A_{sk})$ the root is compared with the symbol L_k, and the end segments are compared with the symbols x_μ, which are encountered in the superposition $L_k(A_1, \dots, A_{sk})$.

It is easy to see that the geometric image of the superposition $L_k(A_1, \dots, A_{sk})$ represents a tree with a separated segment--root [13]. The segments of this tree are compared with the symbols L_k or x_μ, moreover, if in the superposition one object is within another, then the corresponding segments are situated in such a manner that it is possible to indicate the path which passes at first through the first (i. e., corresponding to the outer object) segment and then through the second. Henceforth, the geometric image of the superposition will be called a tree with symbols of (L, x)-tree.

Example. The superposition $L_2(L_1(x_1, x_2, x_1, x_3), x_4, x_3)$ has in a geometric image the (L, x)-tree shown in Fig 2. If all the symbols L_k and x_μ are erased in the (L, x)-tree, then the resulting formation will be called a tree.

1.3. Isomorphism. In the calculation of the number of superpositions we shall need the concept of isomorphism of superpositions of σ, and also the concept of the isomorphism of trees.

Determination. Two superpositions are isomorphous then and only then if they are graphically identical, i. e., on the same places there are the same signs of the alphabet consisting of symbols L_k, x_μ, brackets, and commas.

We shall place in conformance with the tree the word of the alphabet consisting of the symbols L x brackets, and commas. This word is obtained if, in some kind of superposition from σ, the geometric image of which is the tree, we substitute the symbols L_k for L, and x_μ for x. Moreover, the resulting words will be the same regardless of the superposition, the image of which is the given tree, we take.

Determination. Two trees are isomorphous if their corresponding words are identical.

1.4. Index of simplicity. We shall introduce the concept of the index of simplicity which characterizes

sapidly the complexity of the superposition.

Determination. The index of simplicity is called the nonnegative functional determined on the union of the union of the masses σ_0, σ and $\{x_\nu\}$ and equal to zero by the mass $\{x_\nu\}$ is called the index of simplicity.

Examples. The index of simplicity can be determined in the following manner:

(a) the index of simplicity of the superposition is equal to the number of end segments of its corresponding tree (this number is equal to the number of symbols x_ν encountered in the superposition); the index of simplicity of the l-place elementary object from σ_0 is equal to $l-1$, $l=1, 2, \dots$

(b) the index of simplicity of the superposition is equal to the number of all the segments of its corresponding tree (this number is equal to the total number of symbols L_k and x_ν encountered in the superposition: the index of simplicity of any l-place elementary object from σ_0 is equal to l.

(c) the index of simplicity of the superposition is equal to the class of this superposition: on σ_0 it is determined in a random manner.

Let $L_k(, \dots ,)$-be the elementary s_k-place object, $A_1, \dots, A_{s_k}$. $L_k(A_1, \dots, A_{s_k})$--the superpositions from σ, $I[L_k(, \dots ,)]$, $I[A_1], \dots, I[A_{s_k}]$, $I[L_k(A_1, \dots, A_{s_k})]$-the corresponding indexes.

Let the index satisfy the following condition:

$$I[L_k(A_1, \dots, A_{s_k})] \geqslant I[L_k(, \dots ,)] + I[A_1] + \dots + I[A_{s_k}]. \quad (1)$$

It follows from (1) that

$$I[L_k(A_1, \dots, A_{s_k})] \geqslant \sum_{l>1} n_l p_l, \quad (2)$$

where p_l-is the lower edge of the indexes of all the l-place elementary objects which comprise σ_0, n_l- is the number of l-place elementary objects in the superposition $L_k(A_1, \dots, A_{s_k})$.

Frequently, the indexes satisfy a stronger condition than (2):

$$I[L_k(A_1, \dots, A_{s_k})] = p_0 + \sum_{l>1} n_l p_l, \quad (3)$$

where p_0- is a non-negative constant.

The condition (3) is also satisfied by the index of the example (a) (see lemma 3). The index of example (b) also satisfied (3). The index (c) does not satisfy the condition (2) no matter how it is determined for σ_0 (except the case when for σ_0 the index is equal to zero).

Henceforth always, without special reservations, we will assume that the indexes being examined by us satisfy the condition (2).

1.5. Realization. Schannon Function $L(D_h)$. Let $\{f\}$ – be the mass of certain elements. We shall determine the concept of the realization of $\{f\}$ by the superpositions from σ.

We shall compare with each superposition a certain single element f; moreover, let for any element f be a superposition with which it is compared. The superpositions which comprise the prototype f, are called its realizations.

Let D_h – be a random submass of $\{f\}$.

We shall designate by $L(f)$ the minimum index of the superpositions from (σ) which realize f; we shall designate the $\max\limits_{f \in D_h} L(f)$ through $L(D_h)$

In other words, $L(D_h)$ is such a smallest number that any element from D_h can be realized by the superpositions from σ, the index of which does not exceed $L(D_h)$.

$L(D_h)$ characterizes the possibilities of the realization of the elements from D_h.

1.6. Two examples. For illustration of the concept of the superposition, we shall examine two sapid examples.

A. Let P – be the closed class of functions of two-significant algebra of logic, that is a sub mass of functions of the algebra of logic which contains along with any system of functions their random iteration. On the strength of the Post theorem [7], this class has a finite base

$$\varphi_1(x_1, \ldots, x_{t_1}), \ldots, \varphi_r(x_1, \ldots, x_{t_r}), \qquad (4)$$

that is, any function of P is expressed as a formula by the function (4). The given class P can be realized by superpositions. For this, we shall select as a mass of elementary objects the mass $\sigma_0 = \{L_1(, \ldots,), \ldots, L_r(, \ldots,)\}$, where L_i is versus the number $s_i = t_i$, $i = 1, 2, \ldots, r$. We shall assume that the mass $\{x_\mu\}$ has the same capacity as the mass of arguments of functions of the class P. The mass of superpositions of the indicated elementary objects is in mutually well-defined conformance with the mass of all the iterations of the functions (4): if in the superposition we substitute the symbol L_i for φ_i, and the symbol x_μ is replaced by the symbol of the argument x_μ, then we obtain the corresponding iteration. In order to obtain the realization of the elements of P by superpositions from σ, we shall compare for each superposition a function which is determined by the corresponding iteration. Let D_j – be the subclass P, which contains all the functions from j arguments $m = m(j)$ – the number of these functions.

$L(D_j)$, according to 1.5., is equal to such a smallest number that any function of j arguments can be expressed by iterations of the functions (4), the index of which does

not exceed $L(D_j)$ (by index of iteration we understand the index of the corresponding superposition).

B. Let us examine class S of two-pole strongly connected networks (see [8]), closed with respect to the operation R of the replacement of the edge of one network with another network; both possible methods of replacement are solved (see Fig 5) (A two-pole network is a finite graph (see [3]) in which two peaks are marked; these are called poles. The boundary peaks of the subgraph γ of the given network Γ are called peaks common for γ and its supplement, i. e., a subgraph of this network consisting of edges not belonging to γ and their ends and which also comprise the subgraph of the pole. Let p_γ^Γ – be the number of boundary peaks in the subgraph γ of the network Γ The network is called strongly connected if it is connected and the minimum p_γ^Γ taken from all the subgraphs Γ (except the subgraph consisting of the entire network Γ) is not less than two. A strongly connected network is called indivisible (non-separable) if the minimum p_γ^Γ taken with respect to the subgraphs Γ (except the subgraph consisting of the entire network and subgraphs consisting of one edge), is not less than three.). The mass of all the indivisible networks $\{\Gamma_1^{t_1}, \ldots, \Gamma_r^{t_r}\}$, $r \leqslant \infty$ (t_i – is the number of edges in the network $\Gamma_i^{t_i}$), which comprise S, forms the base of S, i. e., any network from S is obtained from the base ones by the application of a finite number of operations R. The base of the class of all the strongly connected networks consists of all the indivisible, strongly connected networks.

Determination. The class, the base of which consists of a finite number of indivisible networks, is called the class with a limited topology.

An example of the class with a limited topology is the class of π- networks. The base of this class consists of three networks shown in Fig 3. By adding to this base a bridge (Fig 4), we obtain another class. Class S, closed with respect to R, can be realized by superpositions. For this purpose, we shall select as a mass of elementary objects the mass

$$\sigma_0 = \{L_1(, \ldots,) L_1'(, \ldots,), \ldots, L_r(, \ldots,), L_r'(, \ldots,)\},$$

where L_i, and L_i' are the same number $s_i = t_i$, $i = 1, \ldots, r$. As a mass of the symbols $\{x_\mu\}$ we shall take the mass from one element x (we will place it in conformance with all the edges). $L_i(, \ldots,)$ and $L_i'(, \ldots,)$ correspond to two possible methods of substituting the network $\Gamma_i^{t_i}$ into another network instead of an edge.

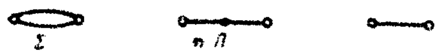

Fig 3

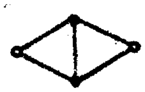

Fig 4

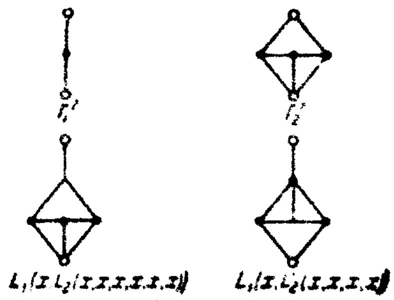

Fig 5

Each superposition from σ describes in a well-defined manner the process of obtaining a certain network from S by the application of the operation R to the base networks (see, for example, Fig 5). Moreover, for one network there can be corresponding different superpositions from σ. However, in any case, σ realizes S (The conformance between σ and S can be described in greater details (see [8]).).

If the base of S includes $\nu_l(S)$ indivisible networks with l edges, then σ_0 includes $2\nu_l(S)$ pieces of l-place objects.

2. Evaluation of $L(D_h)$

2.1. Presentation of the problem. Method of its solution. In the language of the introduced concepts, our problem consists of the evaluation of $L(D_h)$ from below. The idea of an evaluation is the same as in 1, 2, 3 and in various other works. At the beginning (theorem 1), we calculate how many non-isomorphous superpositions which realize class D_h, have an index that does not exceed n. Then we find the asymptotic expression of the upper edge of those n, for which the indicated number of superpositions is less than the number of elements in D_h. This upper edge will give the lower evaluation for $L(D_h)$.

2.2. Certain limitations imposed on the index. The purpose of this note is the evaluation of $L(D_h)$ for random indexes which satisfy the condition (2). In order to obtain a non-trivial result which would find application without the concretization of the concept of realization, we should require that the number of superpositions with an index not in excess of n should be finite for all n. Actually, in the opposite case there would be n_1 – such a smallest number that the number of superpositions, the index of which does not exceed n_1, is infinite. In this case, we cannot be assured that all the elements of any class of D_h are realized by the superpositions with an index not exceeding n_1, and

the following supposition would be correct: regardless of the type of mass of D_k, there will always take place $n_1 > L(D_k)$, and if $n_2 > n_1$, then for any mass of D_k with a sufficiently large number of elements, $L(D_k) > n_2$. Such indexes are not of interest and, for this reason, we shall not examine these (although even for these it would be possible to present corresponding results).

And thus, we will require that the number of superpositions, the index of which does not exceed n, should be finite for all n.

In the case in which the index satisfies the condition (3), the fulfillment of the conditions (5) and (6) is necessary for this:

$$\lim_{l\to\infty} p_l = \infty \quad \text{(if the base } \sigma_0 \text{ is infinite)} \tag{5}$$

$$p_l > 0, \quad l = 1, 2, \ldots \tag{6}$$

Actually, if the base is infinite and (5) is not fulfilled, then all the p_l do not exceed a certain constant p; then for each l there will be a l place object $L_k(\ , \ldots, \ )$ with an index p_l; because of (3), $I[L_k(x_1, \ldots, x_l)] = p_l + p_0 < p + p_0 =$ =const. Consequently, the number of superpositions (even of first class), the index of which does not exceed a certain constant, is infinite.

Let (6) not be fulfilled i. e., $p_l = 0$ for a certain l. For example, let $p_2 = 0$ and $I[L_1(,)] = 0$. All the superpositions: $L_1(x, x)$, $L_1(x, L_1(x, x))$, $L_1(x, L_1(x, L_1(x, x)))$, ... will, because of (3), have the index p_0, i. e., the number of isomorphous superpositions, the index of which is equal to p_0, is infinite.

Since (3) is a specific case of (2) and we wish to examine any indexes which satisfy (2), then (5) and (6) are necessary conditions for the situation in which the number of superpositions, the index of which does not exceed n, is to be infinite for all n.

However, the requirement of $p_l > 0$ is unusually strong. In examining such an important index as the index of the example (2) in paragraph 1.4, we encounter the case of $p_1 = 0$. For this reason, we shall solve the conversion p_1 into zero and replace the condition (6) by the condition (6'):

$$p_l > 0, \quad l = 2, 3, \ldots \tag{6'}$$

But, as follows from the above presentation, we will be compelled to impose a certain limitation on the realization (see condition (7)).

We shall designate by $\sigma^{(r)}$ the mass of those superposition from σ, in which a single-place object is not

encountered in succession more than τ times. Example: the superposition

$$L_1(L_1(L_2(L_3(L_1(x)),\ L_1(x),\ L_1(x),\ L_1(x))))$$

belongs to $\sigma^{(\tau)}$, if $\tau \geqslant 2$. The condition which we impose on the realization is formulated in the following manner:

If class D_h is realized by the superpositions from σ, then there exists such a $\tau > 0$ that it is realized also by the superpositions from $\sigma^{(\tau)}$ (i. e., the protype of each element f from D_h includes at least one superposition from $\sigma^{(\tau)}$). (7)

This condition could be imposed only in an examination of those indexes for which $p_1 = 0$; but it is more convenient to require its fulfillment all the time. It is not excessively burdening and is fulfilled in cases of interest to us.

The conditions (5), (6'), and (7) are sufficient so that the number of superpositions, the index of which does not exceed n is finite for all n (this follows from theorem 1).

Henceforth, we shall always assume that the conditions (2), (5), (6), and (7) are fulfilled.

2.3. Designations. We shall introduce the following designations:

1. k_l — is the number of elementary l-place objects in σ_0; k_l is finite for all l because of 1.1.

2.

$$\rho(n) = \begin{cases} \inf\limits_{l \geqslant 2,\ p_l \leqslant n} \dfrac{p_l}{l-1}, & \text{if } p_1 = 0, \\ \inf\limits_{l \geqslant 1,\ p_l \leqslant n} \dfrac{p_l}{l-1}, & \text{if } p_1 > 0. \end{cases}$$

The number $\rho(n)$ can be called the specific weight of the l-place elementary object. It is obvious that $\rho(n)$ — is a non-increasing function not equal to zero (because of (5) and (6')).

3.

$$\psi(n) = \begin{cases} \sup\limits_{l \geqslant 2,\ p_l \leqslant n} \dfrac{\log_2 k_l}{p_l}, & \text{if } p_1 = 0, \\ \sup\limits_{l \geqslant 1,\ p_l \leqslant n} \dfrac{\log_2 k_l}{p_l}, & \text{if } p_1 > 0. \end{cases}$$

It is obvious that $\psi(n)$ is a definite function, non-decreasing everywhere (because of (5) and (6')).

4. $\tilde{\sigma}^{(\tau)}$ – is a mass of trees corresponding to the superpositions from $\sigma^{(\tau)}$.

5. $d_{\nu,\tau}$ when $\nu > 1$ – is the number of non-isomorphous trees from $\tilde{\sigma}^{(\tau)}$ which have end segments. $d_{1,\tau}$ – is the magnitude of the mass consisting of an empty tree and of trees from $\tilde{\sigma}^{(\tau)}$ with one end segment.

6. $\Delta n, \tau = \sum_{\nu \leq n} d_{\nu,\tau}$.

7. $q_{n,j,\tau}$ – is the number of non-isomorphous (L, x)-trees from $\tilde{\sigma}^{(\tau)}$, which have the following characteristics:

(a) they have an index not greater than n;

(b) the end segments are compared with the symbols x_μ from a fixed mass $\{x_{\mu_1}, \ldots, x_{\mu_j}\}$.

Because of 1.2, $q_{n,j,\tau}$ is equal to the number of superpositions with an index not greater than j and containing not more than n fixed symbols from $\{x_\mu\}$, in which the single-place objects of the index zero are encountered in succession not more than τ times.

Theorem 1.

$$q_{n,j,\tau} < Mj\,(Cj)^{\frac{n}{f(n)}}\,2^{n\varphi(n)},$$

where $C = C(k_1, \tau)$, while M – is a certain absolute constant. We shall preface the proof of the theorem with several lemmas.

Lemma 1 (Compare (12)).

The numbers $d_{\nu,\tau}$ satisfy the recurrent relationships

$$d_{1,\tau} = 1 + \tau; \quad d_{\nu,\tau} = d_{1,\tau} \sum_{\substack{l \geq 2 \\ k_l \neq 0}} \sum_{\substack{\nu_1 + \ldots + \nu_l = \nu \\ \nu_1, \ldots, \nu_l \geq 1}} d_{\nu_1,\tau} d_{\nu_2,\tau} \ldots d_{\nu_l,\tau}. \tag{8}$$

Proof. The trees from $\tilde{\sigma}^{(\tau)}$, which have one end segment are small chains consisting of 1, 2, …, τ segments. Consequently, $d_{1,\tau} = 1 + \tau$.

The number of trees with ν end segments in which $l > 1$ segments come from the root is equal to

$$\delta_{\nu,l} = \sum_{\substack{\nu_1 + \ldots + \nu_l = \nu \\ \nu_1, \ldots, \nu_l \geq 1}} d_{\nu_1,\tau} d_{\nu_2,\tau} \ldots d_{\nu_l,\tau}. \tag{8'}$$

Actually, all such trees can be obtained by joining to l segments of the root star by all possible methods the trees, if only the total number of their end segments were equal to ν. The number of trees with end segments in which 1 segment comes from the root, while the first non-single-place star has $l > 1$ places is obviously equal to

$$\tau\delta_{\nu,l}. \tag{8''}$$

The number of trees with ν end segments in which the first non-single-place star has $l > 1$ places is equal to the sum of (8) and (8''), namely:

$$\delta_{\nu,l} + \tau\delta_{\nu,l} = d_{1,\tau} \cdot \delta_{\nu,l}. \qquad (8''')$$

Summarizing (8''') with respect to all the l, we obtain confirmation of the lemma.

Lemma 2. The number $\Delta_{n,\tau}$ of non-isomorphous trees from $\tilde{\mathfrak{I}}(\tau)$ having not more than n end segments satisfies the inequality

$$\Delta_{n,\tau} < MC_1^n, \qquad (9)$$

where $C_1 = C_1(\tau)$, while M – is a certain absolute constant.

Proof. Let us examine the equation

$$x = \frac{y}{d_{1,\tau}} - \sum_{\substack{l \geq 2 \\ k_l \neq 0}} y^l. \qquad (10)$$

The function $x(y)$ is an analytic function of y when $|y| < 1$, moreover, $\left.\frac{dx}{dy}\right|_{y=0} \neq 0$. For this reason, there is the function

$y = y(x)$, $y(0) = 0$, analytic in a certain neighborhood of the point $x = 0$ and satisfying (10). In this neighborhood

$$y(x) = a_{1,\tau}x + \ldots + a_{\nu,\tau}x^\nu + \ldots \qquad (11)$$

Substituting (11) into (10) and equating the coefficients for x in the right and left portions, we obtain:

$$d_{1,\tau} = a_{1,\tau},$$

$$d_{1,\tau} \sum_{\substack{l \geq 2 \\ k_l \neq 0}} \sum_{\substack{\nu_1 + \ldots + \nu_l = \nu \\ \nu_1, \ldots, \nu_l \geq 1}} a_{\nu_1,\tau} \cdot a_{\nu_2,\tau} \ldots a_{\nu_l,\tau} = a_{\nu,\tau}. \qquad (12)$$

From (12) and (8) we obtain

$$d_{2,\tau} = a_{2,\tau}, \ \ldots, \ d_{\nu,\tau} = a_{\nu,\tau}, \ \ldots$$

The inequality $a_{\nu,\tau} \leq a^1_{\nu,\tau}$, is valid, where $a^1_{\nu,\tau}$ – is the coefficient for x^ν in the dissociation by degrees of x of the solution of the equation

$$x = \frac{y_1}{d_{1,\tau}} - \sum_{l \geq 2} y_1^l.$$

Actually, for $a^1_{\nu,\tau}$ the relationship similar to (12)

is valid, with the only difference that the external sum is taken with respect to all the l, and not just with respect to those where $k_l \neq 0$, as in (12). Special points of the function $y=y(x)$ are the roots of the equations:

$$\left.\begin{aligned} &\frac{d}{dy_1}\left(\frac{y_1}{d_{1,\tau}} - \sum_{l\geq 2} y_1^l\right) = 0, \\ &x = \frac{y_1}{d_{1,\tau}} - \sum_{l\geq 2} y_1^l = 0. \end{aligned}\right\} \tag{13}$$

The smallest with respect to modulus x, which satisfies (13) is

$$x_0 = \frac{1+2d_{1,\tau} - 2\sqrt{d_{1,\tau}(1+d_{1,\tau})}}{d_{1,\tau}}.$$

It can be said (see [9]) that the series $y_1 = \sum_{\nu\geq 1} a^1_{\nu,\tau} x^\nu$ converges when $x = x_0$, and this means that $a^1_{\nu,\tau} x_0^\nu < \mu$, $\nu = 1, 2, \ldots$ Hence, it follows that $d_{\nu,\tau} = a_{\nu,\tau} \leq \ \leq a^1_{\nu,\tau} < MC_1^\nu$, where M – is an absolute constant $C_1 = \frac{1}{x_0}$.

$$\Delta_{n,\tau} = \sum_{\nu\leq n} d_{\nu,\tau} \leq M \sum_{\nu\leq n} C_1^\nu \leq M' C_1^n,$$

where M' – is an absolute constant. Quod erat demonstrandum.

Lemma 3. A tree which numbers n_l pieces of L place stars (or, which is the same, n_l segments, with l segments coming from each of these) has $1+\sum_{l\geq 2} n_l(l-1)$ end segments.

Proof. Actually, the number of all the segments of a tree, including the root, is equal to $1+\sum_{l\geq 1} l\cdot n_l$. The number of non-end segments is equal to $\sum_{l\geq 1} n_l$. For this reason, the number of end segments is

$$1+\sum_{l\geq 1} l n_l - \sum_{l\geq 1} n_l = 1 + \sum_{l\geq 2}(l-1)n_l,$$

which proves the lemma.

Proof of theorem 1.

1. We shall calculate the number of end segments in (L, x)- trees the index of which does not exceed n. These trees are included in the number of trees which satisfy the condition $\sum_{l\geq 1} n_l p_l \leq n$. If such a tree contains an l- plave star, then

$$p_l \leq n, \tag{13'}$$

for this reason, because of $\rho(n)$ for trees, the index of which does not exceed n, the inequality $\rho(n) \leqslant \frac{p_l}{l-1}$. takes place.
The number of end segments of the tree is, because of lemma 3, equal to

$$1+\sum_{l\geqslant 1}(l-1)\,n_l. \tag{14}$$

But

$$1+\sum_{l\geqslant 1}(l-1)\,n_l \leqslant 1+\frac{1}{\rho(n)}\sum_{l\geqslant 1}p_l n_l < 1+\frac{n}{\rho(n)}.$$

Consequently, the trees, the index of which is not greater than n, have no more than $1+\frac{n}{\rho(n)}$ end segments.

2. The number of trees from which by rearrangement of the symbols superpositions are obtained with an index greater than n, is obviously equal to the number of trees the index of which is not greater than n, and, consequently, (lemma 2) does not exceed

$$MC_1^{\frac{n}{\rho(n)}}. \tag{15}$$

where M – is a certain absolute constant while $C_1=C_1(\tau)$.

3. The number of comparisons of j symbols from $\{x_\mu\}$ with end segments of a tree from which by means of a rearrangement of the symbols a superposition is obtained, the index of which is not greater than n, does not exceed

$$j^{1+\frac{n}{\rho(n)}}. \tag{16}$$

4. The number of comparisons of the symbols from $\{L_k\}$ with the internal and root segment of the tree is equal to $\prod_{l\geqslant 1} k_l^{n_l}$. Let $p_l>0$. If the index of the superposition does not exceed n, then $p_l\leqslant n$, and, consequently, log $\frac{\log_2 k_l}{p_l}\leqslant\psi(n)$,

$$\log_2\prod_{l\geqslant 1}k_l^{n_l}\leqslant\psi(n)\sum_{l\geqslant 1}n_l p_l\leqslant n\psi(n), \tag{17}$$

$$\prod_{l\geqslant 1}k_l^{n_l}\leqslant 2^{n\psi(n)}. \tag{18}$$

However, this is so only for those indexes in which $p_1>0$. If $p_1=0$, then the number of comparisons of the symbols L_k, besides the symbols of the single-place objects does not exceed $2^{n\psi(n)}$. Single-place objects, however, can be compared with $k_1^{n_1}$ methods. But from each segment of the tree a small chain can come which contains not more than τ single-place stars. The number of all the segments of the tree, besides single-place ones, is equal to $2n_2+3n_3+\ldots+ln_l+\ldots$

Consequently, $n_1 \leqslant \tau\left(\sum_{l\geqslant 2} l n_l\right) \leqslant 2\tau \sum_{l\geqslant 2}(l-1)n_l \leqslant 2\tau\left(1+\frac{n}{\rho(n)}\right)$ for the superpositions the index of which is not greater than n.

Points 1-4 of the given proof give grounds for asserting that

$$q_{n,j,\tau} < MC_1^{\frac{n}{\rho(n)}} j^{1+\frac{n}{\rho(n)}} 2^{n\phi(n)} k_1^{2\tau\left(1+\frac{n}{\rho(n)}\right)}.$$

This inequality can be recorded in the form of

$$q_{n,j,\tau} < Mj(Cj)^{\frac{n}{\rho(n)}} 2^{n\phi(n)}, \qquad (19)$$

where M – is a certain constant, $C = C(k_1, \tau)$.

The theorem has been demonstrated.

2.4. Evaluation of $L(D_h)$. Let the class D_h have a finite magnitude $m = m(D_h)$. Further, let $j = j(D_h)$ – be such a smallest number that all the elements of D_k can be realized by superpositions which contain not more than j symbols from $\{x_\mu\}$, $\tau = \tau(D_h)$ – is such a smallest number that all the elements of D_h can be realized by superpositions which contain not more than τ single-place objects in succession; τ exists because of the condition (7). Theorem 1 makes it possible for us to establish without labor the following supposition.

Theorem 2. If there is such a sequence of the classes of D_h that $m \to \infty$ and $\frac{\log_2 m}{\log_2 j} \to \infty$ при $h \to \infty$, then

(1) regardless of the value of $\varepsilon > 0$, there is such a $h_0(\varepsilon)$ that $L(D_h) > \; > n_0(1-\varepsilon)$ for all the $h > h_0$, where n_0 – is the greatest of the solutions of the inequality (As n_0 one can take any solution of (20).):

$$n\left(\frac{\log_2 Cj}{\rho(n)} + \phi(n)\right) \leqslant \log_2 m; \qquad (20)$$

(2) the fraction of the elements of the class D_h, which are realized by means of superpositions of an index smaller than $n_0(1-\varepsilon)$, no matter how small, if h is sufficiently great. Theorem 2 says that almost all the elements of the class contain a large number of elements are realized in a very complex manner.

Proof. We shall demonstrate the second assertion of the theorem: the first follows from the second. The number of elements of the class D_h, which are realized by superpositions, the index of which is not greater than n, does not exceed

$$q_{n,j,\tau}, \text{ where } j = j(D_h), \quad \tau = \tau(D_h).$$

Taking into consideration theorem 1 and also the fact that $\rho(n)$ is not increasing while $\phi(n)$ is not decreasing, we obtain:

$$\log_2\left(\frac{q_{n_0(1-\varepsilon),j,\tau}}{m}\right) \leqslant \log_2 M + \log_2 j + (1-\varepsilon)\, n_0 \left(\frac{\log_2 Cj}{\rho(n_0)} + \phi(n_0)\right) - \log_2 m.$$

Because of the fact than n_0 — is the solution of (20) we obtain:

$$\log_2\left(\frac{q_{n_0(1-\varepsilon),j,\tau}}{m}\right) \leqslant \log_2 M + \log_2 j - \varepsilon \log_2 m \to -\infty \qquad (21)$$

when $h\to\infty$ (because when $h\to\infty$ and $m\to\infty$). The inequality (21) shows us that the number of elements from D_h, which are realized by superpositions of an index smaller than $n_0(1-\varepsilon)$, no matter how small in comparison with m, if m is large, which proves the theorem.

Frequently, it is more convenient to use theorem 3 and not 2.

Theorem 3. If there is such a sequence of classes of that

(1) when $h\to\infty$ also $m\to\infty$ and $j\to\infty$;
(2) $\phi(n)\leqslant\psi$, where ψ — is a constant;
(3) $\frac{\log_2 m}{\log_2 j}\to\infty$ when $h\to\infty$,

then

(a) $L(D_h) > n_0(1-\varepsilon)$,

where n_0 — is the greatest solution of the inequality

$$\frac{n}{\rho(n)} \leqslant \frac{\log_2 m}{\log_2 j}, \qquad (22)$$

$\varepsilon\to 0$ when $h\to\infty$;

(b) the fraction of the elements of D_h, which are realized by superpositions with an index smaller than $n_0(1-\varepsilon)$ strives toward zero when $h\to\infty$.

The proof is entirely analogous to the proof of theorem 2 (we are convinced that $\frac{q_{n_0(1-\varepsilon),j,\tau}}{m}\to 0$ when $h\to\infty$). We have the obvious result from theorem 3.

Result. If $\rho(n)\geqslant\rho=$ const, then $L(D_h) > \rho\frac{\log_2 m}{\log_2 j}(1-\varepsilon)$.

Examples of the application of theorem 3:

1. If $p_l \geqslant \rho(l-1)^\alpha$, $0\leqslant\alpha\leqslant 1$, $l=1, 2, \ldots$, $\rho=$ const, then

$$L(D_h) \geqslant \rho\left(\frac{\log_2 m}{\log_2 j}\right)^\alpha (1-\varepsilon),\ \varepsilon\to 0 \text{ by } m\to\infty.$$

Actually,

$$\rho(n) \geqslant \min_{\rho(l-1)^\alpha\leqslant n} \rho(l-1)^{\alpha-1} = \rho\left(\frac{n}{\rho}\right)^{\frac{\alpha-1}{\alpha}}.$$

2. If $p_l > \rho(l-1)^\alpha$, $\alpha > 1$, $\rho =$ const, then

$$L(D_h) \geqslant \rho \frac{\log_2 m}{\log_2 l}(1-\varepsilon).$$

3. If $p_l \geqslant \rho \log_{(r)}(l-1)$ (for large l), where $\rho = \text{const}$, $\log_{(r)}(l-1) = \underbrace{\log_2 \log_2 \dots \log_2}_{r \text{ раз}}(l-1)$, то

$$L(D_h) > \rho \log_{(r)} m (1-\varepsilon).$$

3. Applications of theorems 1, 2, 3

3.1. In the realization of the elements of class D_h we made use only of superpositions in which the single-place function is not encountered in succession more than τ times (condition (7)). The simple lemma demonstrated below gives grounds for asserting that (7) is fulfilled in N- significant logic.

The length of the iteration of the single-place functions will be used to designate the number of functions in it; for example, the length of the iteration $\varphi_1(\varphi_1(\varphi_2(\varphi_1(x))))$ is equal to 4.

Lemma 4. Suppose we have r single-place functions of N-significant logic of $\varphi_1(x), \dots, \varphi_2(x)$. Then there exists such a number τ, that for each iteration of the functions $\varphi_1(x), \dots, \varphi_2(x)$ there will be its equivalent (i. e., which expresses the same function) iteration the length of which does not exceed τ.

Proof. Actually, in the opposite case there would be such an infinite sequence of the numbers $\tau_1 < \dots < \tau_2 < \dots \dots < \tau_i < \dots$ that for each i there will be at least one function expressed by the iterations of the length of τ_i, but not expressed by the iterations of a smaller length. But this is impossible because there is only a finite number of all the single-place functions.

3.2. Application to two-significant algebra of logic. We shall return to the example A, paragraph 1.6. The σ_0- constructed therein is a finite mass; for this reason, the condition (2) of theorem (3) is fulfilled. Further, $\rho(n) \geqslant \rho = \text{const}$. If the index is equal to the number of signs of arguments in the iteration of the base functions, then $\rho = 1$; if, however, it is equal to the number of signs encountered in the iteration of the functions, then $\rho = \frac{1}{s-1}$, where s — is the maximum number of places in the function of the base. We shall designate by m_j the number of functions from j

arguments in the closed class P, and by $L_j(P)$ – the max $L(f)$, taken with respect to the functions from P and depending on the j arguments.

Then (see theorem 3): If $\frac{\log_2 m_j}{\log_2 j} \to \infty$ when $j \to \infty$,

then for any $\varepsilon > 0$ there is such a $j_0(\varepsilon)$ that all the functions from $j > j_0(\varepsilon)$ arguments of class P cannot be realized by means of iterations of the base functions with the index smaller than $\rho \frac{\log_2 m_j}{\log_2 j}(1-\varepsilon)$; by means of iterations of a smaller index, it is possible to express only an infinitely small fraction of the functions from $j > j_0(\varepsilon)$ arguments from P.

Examples.

1. P – is the class of all the functions of the algebra of logic. In this case,

$$m_j = 2^{2^j}, \quad L_j(P) > \rho \cdot \frac{2^j}{\log_2 j}(1-\varepsilon).$$

2. P – is the class of self-dual dunctions. In this case,

$$m_j = 2^{2^{j-1}}, \quad L_j(P) > \rho \cdot \frac{2^j}{2\log_2 j}(1-\varepsilon).$$

3. P – is the class of monotonous functions. In this case,

$$m_j > 2^{2^j \cdot \sqrt{\frac{2}{\pi j}}} \text{ (см. [10])},$$

$$L_j(P) > \sqrt{\frac{2}{\pi}} \rho \cdot \frac{2^j}{j \log_2 j} \cdot (1-\varepsilon).$$

3.3. Application to N-significant logic. In N-significant logic there is no proof of the theorem of the finitiness of the base for closed classes similar to the Post theorem. There is a hypothesis that such a theorem exists.

If such a theorem is valid, then ($L_j(P)$ again designates max $L(f)$ with respect to the functions from j arguments in P):

$$L_j(P) > \rho \frac{\log_2 m_j}{\log_2 j}(1-\varepsilon), \tag{23}$$

where $\rho = \inf_{l>1} \frac{P_l}{l-1} =$ const for any P. If P – is the class of all the functions of N-significant logic, then

$$L_j(P) > \rho \frac{N^j}{\log_N j}(1-\varepsilon),$$

besides, again the fraction of the functions which are realized more simply is infinitely small. However, if the theorem, similar to the Post theorem, is not valid in N-significan logic, then (23) is valid only for classes with a finite base; for classes with an infinite base, the evaluation of $L_i(P)$ is obtained by means of (20) or (22) (see examples of the theorem (3)).

3.4. Application to the theorem of networks. We shall return to the example B from paragraph 1.6. Let us assume that $p_l = l-1$, $l = 1, 2, \ldots$, so that the index is a unit smaller than the number of edges of that network the construction process of which expresses the superposition. Let us assume that the $k_i = 0$, $j = 1$. Then we obtain such a supposition (because of theorem 1):

The number of networks with $n > 1$ edges in the closed class S, the base of which contains $\nu_l(S)$ indivisible networks with l edges does not exceed

$$MC_1^n [\nu_{i(S,n)}]^{\frac{n}{i(S,n)-1}}, \qquad (24)$$

where M and C_1 – are constants, $i(S,n)$ – is the value of l, for which $\max\limits_{l \leq n} [\nu_l(S)]^{\frac{1}{l-1}}$ is attained.

The evaluation for the number s_n of all the strongly connected two-pole networks with n edges stems from [11]:

$$s_n > \left(\frac{C_2 n}{(\log_2 n)^2}\right)^n, \qquad (25)$$

where C_2 – is a constant.

A. Let S – be the class with a limited topology. Then the function $[\nu_{i(S,n)}]^{\frac{n}{i(S,n)-1}}$ is limited by a certain constant and the number of networks with n edges in this class does not exceed MC_2^n, where C_2 – is a const., i.e., the number of networks with n edges in the class with a limited topology is infinitely small in comparison with the number of all the networks with n edges.

B. Let S – be the class of all the two-pole strongly connected networks, ν_l – the number of all the indivisible networks, $i = i(n)$ – the value of l, for which $\max\limits_{l \leq n} \nu_l^{\frac{1}{l-1}}$ is attained.

It follows from (24) and (25) that

$$MC_1^n \nu_i^{\frac{n}{i-1}} > S_n > \left(\frac{C_2 n}{(\log_2 n)^2}\right)^n.$$

Consequently,

$$\nu_i > \left(\frac{C_3 n}{(\log_2 n)^2 C_1 \sqrt[n]{M}}\right)^{i-1}. \qquad (26)$$

Since $i \leqslant n$, then it follows from (26) that

$$\nu_i > \left(\frac{A_i}{(\log_2 i)^2}\right)^i, \qquad (27)$$

where $A =$ const.

Inasmuch as the right portion of the inequality (26) strives toward infinity when $n \to \infty$, then $i \to \infty$ when $n \to \infty$.

Thus, we have proved that there is such an infinite sequence of the numbers $i_1, i_2, \ldots, i_l, \ldots$ that

$$\nu_{i_l} > \left(\frac{A \cdot i_l}{(\log_2 i_l)^2}\right)^{i_l}.$$

We see that a considerable portion of the networks with i_l edges is indivisible.

3.5. Application to the theory of contact circuits. We shall examine class S of networks with a limited topology; we shall compare for each edge one of the contacts $x_1, \bar{x}_1, \ldots, x_j, \bar{x}_j$. We shall obtain class S^* of contact circuits from the j relay, which is naturally called the class of circuits with a limited topology. Class S^* can be realized by superposition by taking σ_0 the same as in the example A from paragraph 1.6, while the mass of symbols is taken as consisting of $2j$ symbols $x_1, x_2, \ldots, x_{2j-1}, x_{2j}$. The number of circuits with a limited topology from the relay j with $n > 1$ contacts, as follows from 3.4, does not exceed $MC_4^n j^{n+1}$, where

$$C_4 = 2C_3 = \text{const}.$$

We shall realize class P of the functions of the algebra of logic by the circuits from S^*. Let $L(f)$ – be the minimum number of contacts necessary for the realization by the circuits from S^* of the function f, $L_j(P) = \max L(f)$, taken with respect to the functions P, and depending on j arguments.

It is not difficult to become convinced of the validity of the following supposition:

if $\frac{\log_2 m_j}{\log_2 j} \to \infty$ when $j \to \infty$, then for any $\varepsilon > 0$ there such a $j_0(\varepsilon)$ that

$$L_j(P) > \frac{\log_2 m_j}{\log_2 j}(1-\varepsilon) \text{ for all } j > j_0(\varepsilon)$$

the fraction of the functions which are realized with a smaller number of contacts is infinitely small when $j \to \infty$.

In particular, if P — are all the functions of the algebra of logic, then

$$L_j(P) > \frac{2^j}{\log_2 j}(1-\varepsilon).$$

For the case in which S^* —is the class of parallel-series circuits, the last evaluation is obtained in the work [1]. We wish to point out that if the class of all the circuits and not the class of circuits with a limited topology is taken as the one being realized, then the following substantially smaller evaluation takes place:

$$L_j(P) > \frac{2^j}{j}(1-\varepsilon).$$

3.6. Suppose we have two two-place functions

$$f(x, y) = x + y; \quad g(x, y) = x \cdot y.$$

Each natural number can be expressed by a formula using these two functions and the constant 1. For example:

$$10 = (\ldots((1+1)+1)\ldots); \quad 10 = ((((1+1)+1)((1+1)+1))+1).$$

Let $L(m)$ — be the smallest number of the signs + and $\cdot$, by means of which it is possible to express all the numbers from 1 to m. In order to obtain from theorem 2 the evaluation of $L(m)$, we assume that $j=1$, $p_2=1$, $k_2=2$, $k_i=0$, if $i \neq 2$.

In this case, it is possible to find a more accurate value of C_1, namely $C_1 = 4$. Here, $\rho(n) = 1$, $\psi(n) = 1$. The inequality (20) takes the form of

$$n \log_2 8 \leqslant \log_2 m.$$

We obtained:

$$L(m) > \frac{1}{3}(1-\varepsilon)\log_2 m,$$

where $\varepsilon \to 0$ when $m \to \infty$; the fraction of the numbers expressed more simply is infinitely small.

It is clear that this is only a very rough evaluation because, among the formulas considered different by us, there are actually many that are equivalent.

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